Kuwait University | **Calculus II : Math 102** | 2 November 2011
Department of Mathematics | **First Examination** | Duration : 90 minutes

---

The use of a calculator of any kind is not allowed. All communication devices including mobile telephones should be switched off. Answer all of the following questions.

1. Let $f(x) = \dfrac{1-2e^x}{1+e^x}$ for $-\infty < x < \infty$.

   (a) Show that f is one-to-one. **(2 pts)**

   (b) Find the domain and range of f^{-1}. **(2 pts)**

   (c) Find a formula for $f^{-1}(x)$. **(2 pts)**

2. Suppose that f is a differentiable one-to-one function on $(-\infty, \infty)$. Let

   $$g(x) = \arctan(f^{-1}(x)).$$

   If $f(-1) = 2$ and $f'(-1) = 3$, then find $g'(2)$. **(2 pts)**

3. Use logarithmic differentiation to find y' where

   $$y = \frac{|x|^{\exp(x^2)}\,(x-3)^{3/5}}{(2^{\sec^{-1} x})\ln|x|}.$$ **(3 pts)**

4. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{x}\left(1 + \frac{x}{\sqrt{4-x^2}}\right)^2 dx$ **(3 pts)**

   (b) $\displaystyle\int \frac{e^x \sinh x}{1 + e^{2x} + 3e^x}\, dx$ **(3 pts)**

5. Find the limit if it exists.

   (a) $\displaystyle\lim_{x\to 0} \frac{\ln(x+e^x)}{e^{\sin x} - \cos x}$ **(4 pts)**

   (b) $\displaystyle\lim_{x\to\infty} \left(\frac{x+3}{\cosh x}\right)^{1/x}$ **(4 pts)**

Kuwait University Department of Mathematics	**Calculus II : Math 102** **First Examination**	2 November 2011 Duration : 90 minutes

SOLUTION

1. (a)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{-x}-2}{e^{-x}+1} = \frac{0-2}{0+1} = -2.$$

$$\lim_{x\to-\infty} f(x) = \frac{1-2(0)}{1+0} = 1.$$

So, since f is continuous, the range of f is $(-2,1)$.

For every $y \in (-2,1)$:

$$y = f(x) = \frac{1-2e^x}{1+e^x} \quad \Rightarrow \quad (1+e^x)y = 1-2e^x \quad \Rightarrow \quad e^x(y+2) = 1-y$$

$\Rightarrow$

$$e^x = \frac{1-y}{y+2} \quad \Rightarrow \quad x = \ln(1-y) - \ln(y+2).$$

So for every y in the range of f there is only one x with $f(x) = y$.

(b) The domain of f^{-1} is $(-2,1)$ and its range is $(-\infty,\infty)$.

(c) $f^{-1}(x) = \ln(1-x) - \ln(x+2)$.

2. $f(-1) = 2 \quad \Rightarrow \quad f^{-1}(2) = -1.$ Therefore

$$\begin{aligned} g'(2) &= \frac{1}{1+[f^{-1}(2)]^2}(f^{-1})'(2) = \frac{1}{1+[f^{-1}(2)]^2}\frac{1}{f'(f^{-1}(2))} = \frac{1}{1+(-1)^2}\frac{1}{f'(-1)} \\ &= \frac{1}{1+(-1)^2}\frac{1}{3} = \frac{1}{6}. \end{aligned}$$

3.

$$\ln|y| = \exp(x^2)\ln|x| + \tfrac{3}{5}\ln|x-3| - (\sec^{-1}x)\ln 2 - \ln|\ln|x||$$

$\Rightarrow$

$$\frac{y'}{y} = \exp(x^2)(2x)\ln|x| + \exp(x^2)\frac{1}{x} + \frac{3}{5}\frac{1}{x-3} - \frac{1}{x\sqrt{x^2-1}}\ln 2 - \frac{1}{\ln|x|}\frac{1}{x}$$

$\Rightarrow$

$$\begin{aligned} y' &= \left[\frac{2x^2\ln|x|+1}{x}\exp(x^2) + \frac{3}{5(x-3)} - \frac{\ln 2}{x\sqrt{x^2-1}} - \frac{1}{x\ln|x|}\right] \\ &\times \frac{|x|^{\exp(x^2)}\,(x-3)^{3/5}}{(2^{\sec^{-1}x})\ln|x|}. \end{aligned}$$

4. (a)

$$\begin{aligned}\int \frac{1}{x}\left(1+\frac{x}{\sqrt{4-x^2}}\right)^2 dx &= \int \left(\frac{1}{x}+\frac{2}{\sqrt{4-x^2}}+\frac{x}{4-x^2}\right) dx \\ &= \ln|x| + 2\arcsin(x/2) - \tfrac{1}{2}\ln(4-x^2) + C.\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{e^x \sinh x}{e^{2x}+3e^x+1} dx &= \frac{1}{2}\int \frac{e^{2x}-1}{e^{2x}+3e^x+1} dx = \frac{1}{2}\int \frac{e^x - e^{-x}}{e^x+3+e^{-x}} dx \\ &= \tfrac{1}{2}\ln(e^x+3+e^{-x}) + C.\end{aligned}$$

5. (a) The limit is indeterminate of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\frac{d}{dx}\ln(x+e^x)}{\frac{d}{dx}(e^{\sin x}-\cos x)} = \lim_{x\to 0} \frac{(1+e^x)/(x+e^x)}{e^{\sin x}\cos x + \sin x} = \frac{(1+1)/(0+1)}{1(1)+0} = 2.$$

So, by L'Hospital's Rule, $\lim\limits_{x\to 0} \dfrac{\ln(x+e^x)}{e^{\sin x}-\cos x} = 2.$

(b)

$$\ln\left(\frac{x+3}{\cosh x}\right)^{1/x} = \frac{1}{x}\ln\left(\frac{x+3}{\cosh x}\right) = \frac{\ln(x+3)}{x} - \frac{\ln(\cosh x)}{x}.$$

$\lim\limits_{x\to\infty} \dfrac{\ln(x+3)}{x}$ and $\lim\limits_{x\to\infty} \dfrac{\ln(\cosh x)}{x}$ are indeterminate of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x+3)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{1/(x+3)}{1} = \lim_{x\to\infty} \frac{1}{x}\frac{1}{1+3/x} = 0\frac{1}{1+3(0)} = 0.$$

$$\begin{aligned}\lim_{x\to\infty} \frac{\frac{d}{dx}\ln(\cosh x)}{\frac{d}{dx}x} &= \lim_{x\to\infty} \frac{\sinh x/\cosh x}{1} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{1-0}{1+0} = 1.\end{aligned}$$

So, by L'Hospital's Rule, $\lim\limits_{x\to\infty} \dfrac{\ln(x+3)}{x} = 0$ and $\lim\limits_{x\to\infty} \dfrac{\ln(\cosh x)}{x} = 1.$

Hence, $\lim\limits_{x\to\infty} \ln\left(\dfrac{x+3}{\cosh x}\right)^{1/x} = 0 - 1 = -1.$

Answer: $\lim\limits_{x\to\infty} \left(\dfrac{x+3}{\cosh x}\right)^{1/x} = e^{-1}.$